

More simple proofs of Sharkovsky's theorem

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1 Introduction

Let f be a continuous map from a compact interval I into itself and let the Sharkovsky's ordering \prec of the natural numbers be defined as follows:

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec 2 \cdot 9 \prec \cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec 2^2 \cdot 9 \prec \cdots \\ \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Sharkovsky's theorem [11], [13] states that (1) if f has a period- m point and if $m \prec n$ then f also has a period- n point; (2) for each positive integer n there exists a continuous map $f : I \rightarrow I$ that has a period- n point, but has no period- m point for any m with $m \prec n$; and (3) there exists a continuous map $f : I \rightarrow I$ that has a period- 2^i point for $i = 0, 1, 2, \dots$ and has no periodic point of any other period. There have been a number of different proofs [1-10, 12-17] of it in the past 30 years providing various viewpoints of this beautiful theorem. The sufficiency of Sharkovsky's theorem is well-known [7] to be a consequence of the following three statements: (a) if f has a period- m point with $m \geq 3$, then f also has a period-2 point; (b) if f has a period- m point with $m \geq 3$ and odd, then f also has a period- $(m+2)$ point; and (c) if f has a period- m point with $m \geq 3$ and odd, then f also has a period-6 point and a period- $(2m)$ point. Note that in (c) we don't require the existence of periodic points of all even periods. Only the existence of period-6 and period- $(2m)$ points suffices. Among these three statements, (a) and (b) are easy to prove. So, if we can find an easy proof of (c), then with the easy counterexamples in [1], [7] or [9], we will have a simple proof of Sharkovsky's theorem. In [7], [8], we present two different proofs of (c). In this note, we present yet another four different proofs of (c) (including one that is a variant of the proof given in [8]) and hence of Sharkovsky's theorem.

In the sequel, let P be a period- m orbit of f with $m \geq 3$, let b be the point in P such that $f(b) = \min P$, and let v be a point in $[\min P, b]$ such that $f(v) = b$.

2 A directed-graph proof of (c)

This directed-graph proof of (c) is different from the one presented in [7]. Let $z_0 = \min\{v \leq x \leq b : f^2(x) = x\}$. Then $\min P = f^2(v) < v < z_0 < \max P$ and $f(x) > x$ for all $v \leq x < z_0$. Since $m \geq 3$ is odd, $\max P$ is also a period- m point of f^2 . Hence $\max\{f^2(x) : \min P \leq x \leq v\} > z_0$. Let $I_0 = [\min P, v]$ and $I_1 = [v, z_0]$. For each $n \geq 0$, the cycle $I_1(I_0)^n I_1$ (with respect to f^2) gives a period- $(n+1)$ point w in $[v, z_0]$ for f^2 such that $f^{2i}(w) < w$ for all $1 \leq i \leq n$. Since $f^{2i}(w) < w < f(w)$ for all $1 \leq i \leq n$, w is a period- $(2n+2)$ point of f . Therefore, f has periodic points of all even periods. This proves (c).

3 A unified non-directed graph proof of (a), (b) and (c)

The proof we present here is a variant of the proof given in [8]. Let z be a fixed point of f in $[v, b]$. Then since $f^2(\min P) > \min P$ and $f^2(v) = \min P < v$, the point $y = \max\{\min P \leq x \leq v : f^2(x) = x\}$ exists and $f(x) > z$ on $[y, v]$ and $f^2(x) < x$ on $(y, v]$. So, y is a period-2 point of f . This proves (a). On the other hand, assume that $m \geq 3$ is odd. Then $f^{m+2}(y) = f(y) > y$ and $f^{m+2}(v) = \min P < v$. So, the point $p_{m+2} = \min\{y \leq x \leq v : f^{m+2}(x) = x\}$ exists and, since $f^2(x) < x$ on $(y, v]$, is a period- $(m+2)$ point of f . This proves (b). We now prove (c). Let $z_0 = \min\{v \leq x \leq b : f^2(x) = x\}$. Then $z_0 \leq z$ and $f(x) > z > x > f^2(x)$ on $[v, z_0]$. If $f^2(x) < z_0$ on $[\min P, y]$, then $f^2(x) < z_0$ on $[\min P, z_0]$. In particular, $f^2([\min P, z_0] \cap P) \subset [\min P, z_0] \cap P$. Since v lies in $[\min P, z_0]$ and since $f^2(v) = \min P$ is in P , we obtain that $f^{2n}(v) < z_0$ for all $n \geq 0$, contradicting the fact that $z_0 \leq z < b = f^m(b) = f^m(f(v)) = f^{m+1}(v) = (f^2)^{(m+1)/2}(v)$. Therefore, the point $d = \max\{\min P \leq x \leq y : f^2(x) = z_0\}$ exists and on (d, y) , we have $f(x) > z \geq z_0 > f^2(x)$. Consequently, $f(x) > z \geq z_0 > f^2(x)$ on (d, z_0) . For each $n \geq 1$, let $c_{2n} = \min\{d \leq x \leq y : f^{2n}(x) = x\}$. Then $d < \dots < c_{2m+2} < c_{2m} < c_{2m-2} < \dots < c_4 < c_2 \leq y$. Furthermore, $f^2(x) < z_0$ on (d, c_2) , $f^4(x) < z_0$ on (d, c_4) , \dots , $f^{2n}(x) < z_0$ on (d, c_{2n}) , \dots , and so on. In particular, for each $n \geq 1$, $f^{2i}(c_{2n}) < z_0$ for all $0 \leq i \leq n-1$. Since $f(x) > z \geq z_0 > f^2(x)$ on (d, z_0) , each c_{2n} is a period- $(2n)$ point of f . Therefore, f has periodic points of all even periods. (c) is proved.

4 A preliminary result

In the previous section, when we prove (a), we obtain a side result. That is, $f^2(v) < v < z < f(v) = b = f^m(b) = f^{m+1}(v)$. So, when $m \geq 3$ is odd, we have $f^2(v) < v < z < (f^2)^{(m+1)/2}(v)$. Surprisingly, these inequalities imply, by Lemma 1 below, the existence of periodic points of all periods for f^2 . However, the existence of periodic points of all periods for f^2 does not necessarily guarantee the existence of periodic points of all *even* periods for f . It only guarantees the existence of periodic points of f with least period $2k$ for each even $k \geq 2$ and least period ℓ or 2ℓ for each odd $\ell \geq 1$. We need a little more work to ensure the existence of periodic points of all even periods for f . By doing some suitable "surgery" to the map f to remove all those periodic points of f of

odd periods j with $3 \leq j \leq m$, we can achieve our goal. We present two such strategies in the next two sections.

Lemma 1. If there exist a point v , a fixed point z of f , and an integer $k \geq 2$ such that either $f(v) < v < z \leq f^k(v)$ or $f^k(v) \leq z < v < f(v)$, then f has periodic points of all periods. In particular, if f has a periodic point of odd period ≥ 3 , then f^2 has periodic points of all periods.

Proof. Without loss of generality, we may assume that $f(v) < v < z \leq f^k(v)$ and f has no fixed points in $[v, z)$. Thus, $f(x) < x$ on $[v, z)$. If $f(x) < z$ for all $\min I \leq x \leq v$, then $f(x) < z$ for all x in $[\min I, z)$. Consequently, $f^i(v) < z$ for each $i \geq 1$. This contradicts the fact that $f^k(v) \geq z$. So, the point $d = \max\{\min I \leq x \leq v : f(x) = z\}$ exists and $f(x) < z$ on (d, z) . Let $s = \min\{d \leq x \leq z : f(x) = z\}$. If $s > d$, then $d < s \leq f(x) < z$ on (d, z) which implies that $f^i(v) < z$ for each $i \geq 1$. This, again, contradicts the fact that $f^k(v) \geq z$. Thus, $s \leq d$. For each positive integer n , let $p_n = \min\{d \leq x \leq z : f^n(x) = x\}$. Then p_n is a period- n point of f . \square

5 The second non-directed graph proof of (c)

Let $z_1 \leq z_2$ be the smallest and largest fixed points of f in $[v, b]$ respectively. Let g be the continuous map on I defined by $g(x) = \max\{f(x), z_2\}$ if $x \leq z_1$; $g(x) = \min\{f(x), z_1\}$ if $x \geq z_2$; and $g(x) = -x + z_1 + z_2$ if $z_1 \leq x \leq z_2$. Then $g([\min I, z_1]) \subset [z_2, \max I]$ and $g([z_2, \max I]) \subset [\min I, z_1]$ and $g^2(x) = x$ on $[z_1, z_2]$. So, g has no periodic points of any odd periods ≥ 3 . Since $m \geq 3$ is odd, for some $1 \leq i \leq m - 1$, both $f^i(b)$ and $f^{i+1}(b)$ lie on the same side of $[z_1, z_2]$. Let k be the smallest among these i 's. If k is odd then $g^2(v) = f^2(v) = f(b) = \min P < v < z_1 = g^{k+3}(v)$ and if k is even then $g^2(v) = f^2(v) = f(b) = \min P < v < z_1 = g^{k+2}(v)$. In either case, we have $g^2(v) < v < z_1 = (g^2)^n(v)$ for some $n \geq 2$. By Lemma 1, g^2 has periodic points of all periods. Since g has no periodic points of any odd periods ≥ 3 , g has periodic points of all even periods ≥ 4 which obviously are also periodic points of f with the same periods. This proves (c).

6 The third non-directed graph proof of (c)

We may assume that v is the largest point in $[\min P, b]$ such that $f(v) = b$. Let $z_1 \leq z_2$ be the smallest and largest fixed points of f in $[v, b]$ respectively. Without loss of generality, we may assume that $I = [\min P, \max P]$ and $f(x) = x$ for all $z_1 \leq x \leq z_2$. By Lemma 1, we may also assume that f has no fixed points in $[\min P, v] \cup [b, \max P]$. If f has a period-2 point \hat{y} in $[v, z_1]$, then $z_2 < f(\hat{y}) < b$. In this case, let $t = \hat{y}$ and $u = f(\hat{y})$ and let h be the continuous map from I into itself defined by $h(x) = f(x)$ for x not in $[t, u]$ and $h(x) = -x + t + u$ for x in $[t, u]$. Otherwise, let $t = v$ and $u = z_2$ and let $h(x) = f(x)$ for x in I . Consequently, if Q is a period- k orbit of h with $k \geq 3$, then $[\min Q, \max Q] \supset [t, u]$. Let $r = \min\{\max Q : Q \text{ is a period-}m \text{ orbit of } h\}$. Then r is a period- m point of h . Let R denote the orbit of r under h . By (b), $(\min R, \max R)$ contains a period- $(m + 2)$ orbit W of h . Let \hat{h} be the continuous map from I into itself defined by $\hat{h}(x) = \min W$ if $h(x) \leq \min W$; $\hat{h}(x) = \max W$ if $h(x) \geq \max W$; and $\hat{h}(x) = h(x)$ elsewhere. Then

clearly \hat{h} has no period- m points. By (b), \hat{h} has no period- n points for any odd n with $3 \leq n \leq m$. Since \hat{h} has the periodic orbit W of odd period $m+2$, by Lemma 1 and the above, \hat{h} has period- $(2j)$ points for each odd j with $3 \leq j \leq m$ which obviously are also periodic points of f with the same periods. This proves (c).

7 A proof of Sharkovsky's theorem

For the sake of completeness, we include a proof of Sharkovsky's theorem which is slightly different from those in [7], [8]. If f has period- m points with $m \geq 3$ and odd, then by (b) f has period- $(m+2)$ points and by (c) f has period- $(2 \cdot 3)$ points. If f has period- $(2 \cdot m)$ points with $m \geq 3$ and odd, then f^2 has period- m points. According to (b), f^2 has period- $(m+2)$ points, which implies that f has either period- $(m+2)$ points or period- $(2 \cdot (m+2))$ points. If f has period- $(m+2)$ points, then in view of (c) f also has period- $(2 \cdot (m+2))$ points. In either case, f has period- $(2 \cdot (m+2))$ points. On the other hand, since f^2 has period- m points, by (c) f^2 has period- $(2 \cdot 3)$ points, hence f has period- $(2^2 \cdot 3)$ points. Now if f has period- $(2^k \cdot m)$ points with $m \geq 3$ and odd and if $k \geq 2$, then $f^{2^{k-1}}$ has period- $(2 \cdot m)$ points. It follows from what we have just proved that $f^{2^{k-1}}$ has period- $(2 \cdot (m+2))$ points and period- $(2^2 \cdot 3)$ points. It follows that f has period- $(2^k \cdot (m+2))$ points and period- $(2^{k+1} \cdot 3)$ points. Consequently, if f has period- $(2^i \cdot m)$ points with $m \geq 3$ and odd and if $i \geq 0$, then f has period- $(2^\ell \cdot m)$ points for each $\ell \geq i$. It is clear that f^{2^ℓ} has period- m points. By (a), f^{2^ℓ} has period-2 points. This implies that f has period- $2^{\ell+1}$ points whenever $\ell \geq i$. Finally, if f has period- 2^k points for some $k \geq 2$, then $f^{2^{k-2}}$ has period-4 points. Again by (a), $f^{2^{k-2}}$ has period-2 points, ensuring that f has period- 2^{k-1} points. This proves the sufficiency of Sharkovsky's theorem.

For the rest of the proof, it suffices to assume that $I = [0, 1]$. Let $T(x) = 1 - |2x - 1|$ be the tent map on I . Then for each positive integer n the equation $T^n(x) = x$ has exactly 2^n distinct solutions in I . It follows that T has finitely many period- n orbits. Among these period- n orbits, let P_n be the one with the smallest $\max P_n$ (or the one with the largest $\min P_n$). For any x in I let $T_n(x) = \min P_n$ if $T(x) \leq \min P_n$, $T_n(x) = \max P_n$ if $T(x) \geq \max P_n$, and $T_n(x) = T(x)$ elsewhere. It is then easy to see that T_n has exactly one period- n orbit (i.e., P_n) but has no period- m orbit for any m with $m \prec n$ in the Sharkovsky ordering. Now let Q_3 be the unique period-3 orbit of T of smallest $\max Q_3$. Then $[\min Q_3, \max Q_3]$ contains finitely many period-6 orbits of T . If Q_6 is the one of smallest $\max Q_6$, then $[\min Q_6, \max Q_6]$ contains finitely many period-12 orbits of T . We choose the one, say Q_{12} , of smallest $\max Q_{12}$ and continue the process inductively. Let $q_0 = \sup\{\min Q_{2^i \cdot 3} : i \geq 0\}$ and $q_1 = \inf\{\max Q_{2^i \cdot 3} : i \geq 0\}$. Let $T_\infty(x) = q_0$ if $T(x) \leq q_0$, $T_\infty(x) = q_1$ if $T(x) \geq q_1$, and $T_\infty(x) = T(x)$ elsewhere. Then it is easy to check that T_∞ has a period- 2^i point for $i = 0, 1, 2, \dots$ but has no periodic point of any other period. This completes the proof of Sharkovsky's theorem.

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